2201. Show that $f(x) = x^3 + 1$ is neither an even nor an odd function.

2202. State, with a reason, whether the following claims are true or false:

(a) "If quadrilaterals have the same four angles, then they are similar."

(b) "If kites have the same four angles, then they are similar."

2203. Show that, when $4x^3 - 19x^2 + 24x - 12$ is divided by $(x^2 - 4x + 3)$, the remainder is non-zero.

2204. The equations $f(x) = 0$ and $g(x) = 0$, where f and g are cubic functions, have the same solution set S. The equation $f(x) = g(x)$ is denoted E. State, with a reason, whether the following claims hold:

(a) "E has solution set S",

(b) "the solution set of E contains S",

(c) "the solution set of E is a subset of S".

2205. A diving board is set up as below, shown in side view. The small triangles represent supports which are exerting reaction forces on a uniform beam of mass m kg.

Show that, if the lower support divides the beam $1 : 3$, then the total magnitude of the reaction forces exerted by the supports is $3mg$ N.

2206. The area enclosed by the curves $y = k - x^2$ and $y = x^2$ is $64\sqrt{3}$. Determine the value of k.

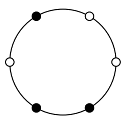
2207. Show that the parametric equations $x + y = \sin t$, $x - y = \cos t$ define a circle in the (x, y) plane, giving its centre and radius.

2208. A first-principles limit is set up as

$$\frac{dy}{dx} = \lim_{p,q \to x} \frac{p^3 - q^3}{p - q}.$$

By factorising, prove that $\dfrac{dy}{dx} = 3x^2$.

2209. Three black and three white beads are threaded onto a bracelet.

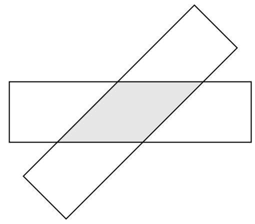


If rotations and reflections of the bracelet are not counted as distinct, show that there are only three possible arrangements.

2210. Show that $\dfrac{d}{d\theta}(\sec^2\theta - \tan^2\theta) = 0$.

2211. Sketch $\sqrt{y} = \sqrt{1 - x^2}$.

2212. Two congruent rectangles, each of area 4, are placed at 45° to each other, both centred on the same point. The area of overlap is $\sqrt{2}$.



Find the dimensions of the rectangles.

2213. Make b the subject of $a = \dfrac{b^2}{2b - 2}$.

2214. Without a calculator, evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \sec^2 2x \, dx$.

2215. The velocity of a particle is given, over the domain $t \in [0, 5]$, by $v = 10t - 3t^2$.

(a) Show that the average velocity is 0.

(b) Find the average speed.

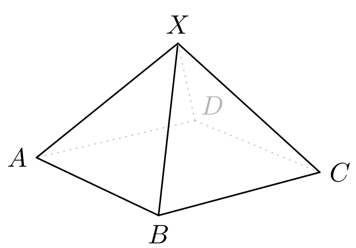
2216. Find the equation of the angle bisector of the lines $y = \sqrt{3}x + 3$ and $y = \frac{\sqrt{3}}{3}x + 1$.

2217. Four circles C_i are defined, for $i = 1, 2, 3, 4$, by

$$\left(x - \cos\frac{\pi i}{2}\right)^2 + \left(y - \sin\frac{\pi i}{2}\right)^2 = 1.$$

Sketch the circles on a single set of axes.

2218. A square-based pyramid is shown below.



If each of the five faces of the pyramid is randomly and independently coloured either blue or yellow, determine the probability that no two adjacent faces are coloured blue.

2219. A variable X has distribution $B(n, \frac{1}{3})$. For some n, $\mathbb{P}(X = 1) = \mathbb{P}(X = 2)$. Using an algebraic method, determine n.

2220. A student writes: "Friction always acts to oppose motion. So, when a car is accelerating forwards, friction must be acting backwards on the car."

Explain whether this is correct.

2221. A quadratic is defined as $f(x) = ax^2 + bx + c$, for constants $a, b, c \in \mathbb{R}$ with $a \neq 0$. Prove that, for some $p \in \mathbb{R}$, $f(p - x) \equiv f(p + x)$.

2222. Assuming $\frac{d}{dx}e^x = e^x$, prove that $\frac{d}{dx}a^x \equiv \ln a \cdot a^x$.

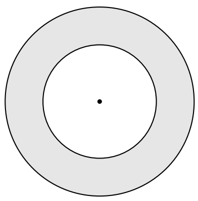
2223. True or false?

(a) $y = \dfrac{1}{1 + x^2}$ has a vertical asymptote,

(b) $y = \dfrac{1}{1 + x + x^2}$ has a vertical asymptote,

(c) $y = \dfrac{1}{1 + 2x + x^2}$ has a vertical asymptote.

2224. An *annulus* is defined as a ring whose boundary consists of two concentric circles, radii $r < R$.



A particular annulus has area 25π. A chord is drawn to the outer circle, which is tangent to the inner circle. Find the length of this chord.

2225. Write $9x^2 - 18x + 7$ in terms of $(3x - 1)$.

2226. Prove that the total surface area of a right-circular cone, in terms of the radius r and height h, is

$$A = \pi r\left(r + \sqrt{r^2 + h^2}\right).$$

2227. A chemist sets up a two-tail binomial hypothesis test, in order to assess allergic reactions to a new antiviral drug. A sample of size 100 is taken. The null hypothesis is $H_0 : p = 0.042$, with p defined as the probability that a member of the population has an allergic reaction to the drug. The chemist sets the significance level at 1%.

(a) Explain why the definition of p refers to the population, rather than the sample.

(b) Write down the alternative hypothesis.

(c) Assuming the null hypothesis, write down the expected number of allergic reactions in the sample.

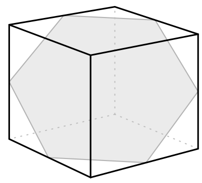
(d) The p-value for the sample is calculated to be 0.0288. Give the conclusions of the test.

2228. Explain, with reference to a sketch of $y = x^2$, why the following holds for any positive constant k:

$$\int_0^k x^2\,dx + \int_0^{k^2} \sqrt{y}\,dy = k^3.$$

2229. A quadratic function f is invertible over each of the domains $\{x : x \leq k\}$ and $\{x : x \geq k\}$. Prove that $y = a(x - k)^2 + b$, for some constants a, b.

2230. The diagram shows a cube of unit side length, and a regular hexagon joining the midpoints of six of the cube's edges.



Find the area of the hexagon.

2231. Show that the distance between the line $y = -x$ and the circle $x^2 - 8x + y^2 - 12y + 44 = 0$ is $3\sqrt{2}$.

2232. If $a^4 b - 4a^2\sqrt{b} + 4 = 0$, write b in terms of a.

2233. A polynomial function f is defined over $\mathbb{R}$. You are given that, for any two constants $a < b \in \mathbb{R}$,

$$\int_a^b f(x)\,dx = \int_a^b |f(x)|\,dx.$$

Explain what this says about the graph $y = f(x)$.

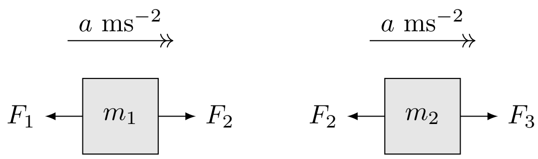
2234. On a single set of axes, sketch $x^2 + k^2y^2 = 1$ in the cases when

(a) $k = 1$,

(b) $k = \frac{1}{2}$,

(c) $k = 0$.

2235. Two objects are modelled with the following force diagrams:



(a) Show that $F_3 - F_1 = (m_1 + m_2)a$.

(b) Show that $F_2(m_1 + m_2) = F_1 m_2 + F_3 m_1$.

2236. Show that the volume of a triangular prism, all of whose edge lengths are l, is given by $\frac{\sqrt{3}}{4}l^3$.

2237. Two samples $\{x_i\}$ and $\{y_i\}$, both with mean zero and size n, have variances s_x^2 and s_y^2. These are then combined to form a sample $\{z_i\}$ of size $2n$, with variance s_z^2. Prove that

$$s_z^2 = \frac{1}{2}\left(s_x^2 + s_y^2\right).$$

2238. Solve $2\log_3 x + \log_3(x+1) = \log_9 144$.

2239. The first three terms of a GP are given by a, b, c. The first three terms of an AP are given by a^2, b^2, c^2. Show that the common ratio of the GP is $r = \pm 1$.

2240. Divide $15x^4 - 8x^3 - 12x^2 + 9x + 6$ by $(3x+2)$.

2241. Using an identity, find the Cartesian equation of the following curve defined parametrically:

$$x = 5 + \sin 2t,$$
$$y = 2 + 3\cos 2t.$$

2242. The probabilities of the variable $X \sim N(0,1)$ are being generated numerically using a rectangle rule, where the top-left hand corner of the rectangle is placed on the bell curve. State, with a reason, whether the following probabilities will be over or underestimated, and how the percentage errors compare in each case.

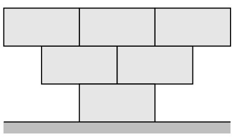
(a) $\mathbb{P}(-1.5 < X < -0.5)$,

(b) $\mathbb{P}(0.5 < X < 1.5)$,

(c) $\mathbb{P}(-0.5 < X < 0.5)$

2243. Find the equation of the line through $(p-1, p+1)$ and (p^2-1, p^2+1).

2244. Six identical, smooth, uniform bricks are stacked as shown, with the lowest resting on flat ground.



Show carefully that this arrangement of bricks can not remain in equilibrium, specifying which bricks will start to topple.

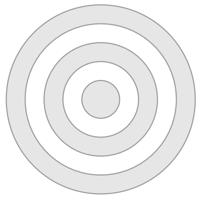
2245. The function $\mathrm{n}(A)$, where A is a set containing a finite number of elements, gives, as an output, the number of elements in A. Prove that, if A and B are such sets,

$$\mathrm{n}(A \cup B) = \mathrm{n}(A) + \mathrm{n}(B) - \mathrm{n}(A \cap B).$$

2246. If $\dfrac{d}{dx}(xy) = 1$, find $\dfrac{dy}{dx}$ in terms of x and y.

2247. If $y = x + 1$, express $x^3 + 3x^2 + 3x + 6$ in simplified terms of y.

2248. An archery target consists of five concentric rings, defined by the regions $(i-1)^2 \le x^2 + y^2 < i^2$ for $i = 1, ..., 5$. The rings score $S = 6 - i$.



Show that, if probability is proportional to area, then the mean score per arrow is 2.2.

2249. Describe the single transformation which takes the graph $y = e^x$ onto the graph $y = 2 - e^x$.

2250. Verify that the following implication holds:

$$y = (x+1)^3\left(\tfrac{1}{2}x^2 + x + 1\right)$$
$$\implies \frac{dy}{dx} - \frac{3y}{x+1} = (x+1)^4.$$

2251. By rewriting in the form $R\cos(\theta - \alpha)$, find all $\theta \in [\pi, \pi]$ which satisfy $\sqrt{3}\cos\theta + \sin\theta = 1$.

2252. Find and classify any and all stationary points of the following curve, and hence sketch it for $x \ge 0$:

$$y = \frac{e^x + x}{e^x}.$$

2253. By factorising, solve the equation $6x^2 + |x| = 1$.

2254. Three cards are dealt from a standard deck. State which, if either, of the following events has the greater probability:

① a queen, then a queen, then a king,

② a queen, then a king, then a queen.

2255. State, with a reason, whether the following are valid implications:

(a) $a > b \implies a^2 > b^2$,

(b) $a > b \implies a^3 > b^3$,

(c) $a > b \implies a^4 > b^4$.

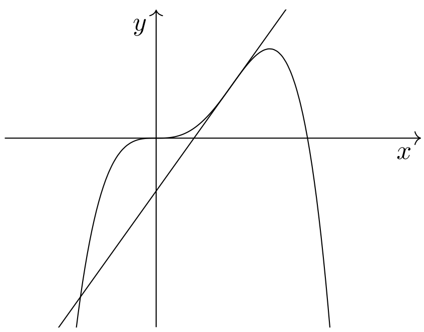
2256. Prove that the product of three consecutive odd integers is divisible by 3.

2257. Given that the sum $1 + \frac{1}{2} + \frac{1}{3} + ...$ diverges, prove that the sum $\log_2 x + \log_4 x + \log_8 x + ...$ diverges for any $x > 1$.

2258. Five sets of bivariate data (x_i, y_i), each with mean $(0,0)$, are plotted on scatter diagrams, and curves of best fit are drawn. Each curve fits its set of data very closely. For each set of data, state, with a reason, whether, as measured by r, there will be correlation, and, if so, whether it will be strong or weak and positive or negative.

(a) $y = 2.713x$,
(b) $y = 0.883x^2$,
(c) $y = -4.155x$,
(d) $y = 0.512x^3$,
(e) $y = -5.125x^3$.

2259. A curve and a line are given by $y = 2x^3 - x^4$ and $y = 2x - 1$. These are shown the diagram:



(a) Show that they are tangent at $(1,1)$.
(b) Show that $(1,1)$ is a point of inflection.
(c) Show that they enclose an area of $\frac{8}{5}$.

2260. In units of radians, find the limit, as $n \to \infty$, of the mean of the interior angles of an n-gon.

2261. Rearrange $px + q\sqrt{x} + r = 0$ to make x the subject.

2262. By splitting into two geometric series, evaluate

$$\sum_{n=1}^{\infty} \frac{2^n + (-2)^n}{4^n}.$$

2263. Prove that, for any function f and constants a, b, the curves $y = \mathrm{f}(x)$ and $y = \mathrm{f}(x) + ax + b$ have the same set of x values at which they are inflected.

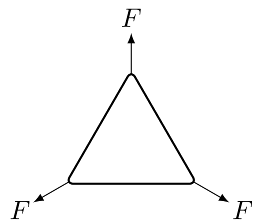
2264. A circle passes through the points $A : (3,8)$ and $B : (10,9)$, and is tangent to the x axis.

(a) Explain why the centre of the circle must lie on the line $y = 54 - 7x$.
(b) Find expressions for the squared distance from the point $(p, 54 - 7p)$ to
 i. the x axis,
 ii. the point $(3,8)$.
(c) By equating the above, solve to find the two possible pairs of coordinates of the centre.

2265. Prove that, if $T_n = \frac{1}{2}n(n+1)$, then

$$T_n + T_{n-1} = (T_n - T_{n-1})^2.$$

2266. A loop of smooth, light string is pulled taut by three coplanar forces each of magnitude F. Under the action of these forces, it forms an equilateral triangle:



Show that the tension in the string is $\frac{\sqrt{3}}{3}F$.

2267. Variables x and y have constant rates of change

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b.$$

Find $\frac{d}{dt}(x+y)^2$ in terms of x, y, a, b.

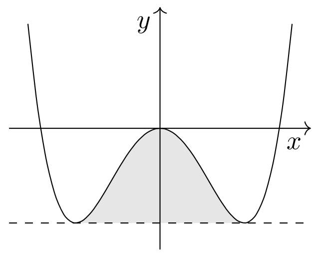
2268. A separable DE, with initial conditions $x = \ln 2$ at $t = 0$, has been manipulated algebraically to give the following:

$$\int e^x \, dx = \int e^t \, dt.$$

Find x when $t = 3\ln 2$.

2269. The interior angles of a pentagon are in AP. Find all possible values for the smallest angle, giving your answer in radians.

2270. The graph below shows $y = \frac{1}{10}x^4 - x^2$, and a line that is tangent to it twice.



Show that the area of the shaded region is $\frac{8}{3}\sqrt{5}$.

2271. If $y = \sqrt{\cos 3x}$, find and simplify $\frac{dy}{dx}$.

2272. For the rule $\mathrm{h} : x \mapsto \frac{2x+1}{2x-1}$, find h^{-1}.

2273. Prove that the largest circle which can be inscribed in a rhombus has radius $r = \frac{1}{2}l \sin\phi$, where l is the side length and ϕ any interior angle.

2274. (a) Using the binomial expansion, write $(\sqrt{3}-2)^5$ in the form $a\sqrt{3}+b$, where $a, b \in \mathbb{Z}$.

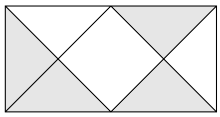
(b) Hence, show that $\sqrt{3} \approx \frac{362}{209}$.

2275. One of the following statements is true; the other is not. Prove the one and disprove the other.

(a) $x \sec x = 0 \implies x = 0$,

(b) $x \cot x = 0 \implies x = 0$.

2276. The seven regions of the schematic map below are shaded, or left blank, at random. In the example, four have been shaded.



Find the probability that, if precisely four are shaded, then no two shaded regions share a border.

2277. Show that the equation of the tangent to the curve $y = \sqrt{1-x^2}$ which passes through the point $(2, 0)$ is $y\sqrt{a} + x = b$, where $a, b \in \mathbb{N}$ are to be found.

2278. A sample of size n has $\sum x = -160$, $\sum x^2 = 1228$, and variance exactly 1.2024. Solve to find n.

2279. Prove that, if a polynomial function is everywhere convex, then it has at most two roots.

2280. Prove the following formula, concerning changes in the side lengths (a, b, c) of a variable triangle whose vertices are at $(0, 0)$, $(a, 0)$, $(0, b)$:

$$a\frac{da}{dt} + b\frac{db}{dt} = c\frac{dc}{dt}.$$

2281. A statistician programs a function to answer the question: "What is the standard deviation of this sample?" The domain is the set of all possible samples, and the codomain is $\mathbb{R}$.

(a) Write down the range.

(b) The domain is now restricted to samples of size 2 with mean zero. The codomain is restricted to $\mathbb{R}^+$. Explain whether this version of the function is invertible.

2282. Show that the curve $y = (e^x - 1)^3$ has a stationary point of inflection.

2283. A packing case of mass m is being hauled up a rough ramp of inclination θ with a light rope and a winch. The coefficient of friction is μ, and the winch moves the packing case at constant speed. Find the tension in the rope, in terms of m, g, μ, θ.

2284. Show that $\displaystyle\int_6^7 \frac{4}{5-x}\,dx = \ln \frac{1}{16}$.

2285. If $x = \cos 2u$, show that $4\dfrac{du}{dx} + \sec u \operatorname{cosec} u = 0$.

2286. By finding derivatives, or otherwise, show that the curve $y = \left(x^2-1\right)^2$ has two local minima and one local maximum.

2287. Using a tree diagram, or otherwise, explain why

$$\mathbb{P}(B \mid A) \equiv \frac{\mathbb{P}(A \mid B)\,\mathbb{P}(B)}{\mathbb{P}(A \mid B)\,\mathbb{P}(B) + \mathbb{P}(A \mid B')\,\mathbb{P}(B')}.$$

2288. The straight line passing through $P : (a, b)$ and $Q : (c, d)$ can be written, in terms of parameter λ,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \lambda \begin{pmatrix} c-a \\ d-b \end{pmatrix}.$$

(a) Describe the points with

i. $\lambda = 0$,

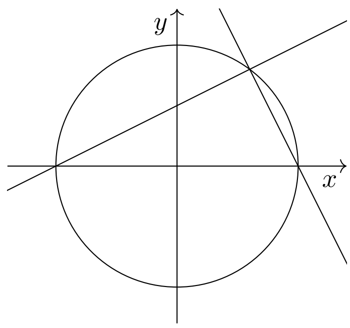
ii. $\lambda = 1$,

iii. $\lambda = \frac{1}{2}$.

(b) Find, in simplified terms of a, b, c, and d, the position vector of the point that divides PQ in the ratio $5 : 3$.

2289. Sketch the curve $\ln x = 1 + 2\ln y$.

2290. The vertices of a triangle lie on the unit circle. Two of its sides are described by the lines $y = 0$ and $2x + y = 2$.



Determine the equation of the straight line which contains the third side.

2291. Use a double-angle formula to show that

$$\int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{\pi}{4}.$$

2292. A hand of two cards is dealt from a standard deck. Given that at least one card is a heart, find the probability that both cards are hearts.

2293. Show that the curves $xy = 1$ and $x^2 + y^2 = 2$ are tangent to each other, and sketch them.

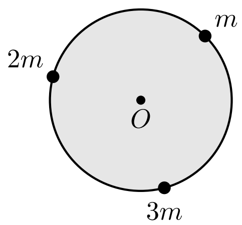
2294. Solve $\sec^2 x - \tan x - 1 = 0$ for $x \in [0, 2\pi)$.

2295. A differential equation is given as

$$\frac{dy}{dx} = x + y + 1.$$

By substituting $y = mx + c$, find the equation of the line which satisfies the DE.

2296. A Catherine wheel has three fireworks mounted on it, with masses as shown below. The fireworks are affixed symmetrically, and the wheel is free to spin in a vertical plane around O.



Show that, if the fireworks of mass m and $3m$ are on the same vertical line, then the wheel will rest in equilibrium.

2297. Solve $\left(\sqrt{x} + \sqrt{x-1}\right)^3 - \left(\sqrt{x} - \sqrt{x-1}\right)^3 = 0$.

2298. Shade the region of the (x, y) plane which satisfies both of the following inequalities:

$$y > |x|, \qquad x^2 + y^2 \leq 4.$$

2299. Separate the variables in the following differential equation, writing it in the form $\mathrm{f}(y)\frac{dy}{dx} = \mathrm{g}(x)$ for some functions f and g:

$$\frac{dy}{dx} + \sin(x + y) = \sin(x - y).$$

2300. A function f is defined over $\mathbb{R}$ as $\mathrm{f}(x) = x^n - 1$, where the index n is chosen at random from the set $\{1, 2, 3, 4, 5\}$. Find the probability that f has input values for which it is decreasing.

——— End of 23rd Hundred ———